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Quasiinterpolant Splines on the Unit Circle

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It is well known that complex analysis has various applications in applied sciences. But for many physical systems, we cannot get the explicit form of the solutions and have to construct approximating functions from the given conditions. We know that a function which maps conformally a simply connected domain D onto the unit disc U satisfies an integral equation of the first kind on the boundary of D [7, 10]. If we solve the integral equation by numerical methods, we obtain a finite set of boundary values of the approximating mapping function; therefore the problem is how to construct an approximating function from the given data.

It seems natural to construct an approximating function using interpolating Lagrange polynomials. But Fejér [6] pointed out that "There is a function $f(z)$ regular in the open disc $|z| < 1$ and continuous in the closed disc $|z| \leqslant 1$ such that the sequence $\{\mathscr{L}_n(f, z)\}_1^n$ of Lagrange interpolating polynomials for equidistant interpolating points on $|Z| = 1$ diverges at the point $Z = 1$; more exactly, $\overline{\lim}_{n\to\infty} \mathscr{L}_n(f, 1) = \infty$."

If an approximation to an analytic function f is required to high accuracy, then we suggest using complex spline functions. Applying the Poisson integral formula, we can construct a complex harmonic spline which gives a good approximation to an analytic function on the closed unit disc $\bar{U}$ [4].

First we have to search for a complex spline which approximates the analytic function on the unit circle Γ.

We could use the interpolating complex splines with equidistant knots. The existence and uniqueness of such splines are proved by many authors [1, 9, 11]. However, if the function to be approximated oscillates rapidly on some parts of Γ (or its image curve has large curvature on some parts), we would like to be able to adjust the knots of the interpolating spline function. The existence of an interpolating spline with arbitrarily spaced knots is proved only for the case $n \leqslant 3$ [2].

Here we introduce the so-called complex quasiinterpolating splines which

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give an efficient approximation of high accuracy to an analytic function on Γ.

Our approach is related to the papers of de Boor and Fix [5], and Lyche and Schumaker [8]. However, in comparison with the real case, some parts of the proofs must be changed, because some theorems and techniques used in the real case are unavailable in complex analysis.

1. Notations and Definitions

Let Γ be the unit circle, $\Delta: z_1,..., z_M$ be points arranged on Γ in counterclockwise order, and $\mathscr{S}_n(\Delta)$ denote the family of complex splines of degree n with knots Δ. If $S \in \mathscr{S}_n(\Delta)$, then S satisfies the conditions: (i) $S \in \pi_n$ on γ_j, $j=\overline{1, M}$, and (ii) $S \in C^{n-1}(\Gamma)$, where π_n denotes the family of polynomials of degree n and γ_j denotes the circular arc $\widehat{z_j z_{j+1}}$.

Let $I_{a,b}$ denote a circular arc, with end points z_a, z_b; $z_a \neq z_b$. If a point z runs in counterclockwise order from z_a to z_b, then z describes the arc $I_{a,b}$. Suppose t_1, t_2 belong to $I_{a,b}$; then $t_1 \ll t_2$ means the point z runs in counterclockwise order starting from point z_a, meets point t, first, then point t_2. In such a manner, we can also write $t_2 \gg t_1$.

Evidently we can distinguish the order of any two points t_1, t_2 on $I_{a,b}$, but this is possible only if $z_a \neq z_b$; in other words, $I_{a,b}$ cannot be a contour. For S, $z \in I_{i,i+n+1}$ $(=\widehat{z_i z_{i+n+1}})$, we define

$$
\begin{aligned}
(S-z)_+^l &= (S-z)^l, && S \gg z, \\
&= 0, && S \ll z \quad \text{or} \quad S = z,
\end{aligned}
$$

where l is any non-negative integer. For any positive integer k, we define the basic spline functions of order k as follows:

$$
\begin{aligned}
N_{i,k}(z) &= (z_{i+k} - z_i)[z_i,..., z_{i+k}]_S (S-z)_+^{k-1}, && z \in I_{i,i+k}, \\
&= 0, && z \in \Gamma \backslash I_{i,i+k},
\end{aligned}
$$

$i=\overline{1, M}$. Here $[z_i,..., z_{i+k}]f$ is the divided difference of the kth order of f with respect to points $z_i,..., z_{i+k}$. (If $i+k=M+j$, $j \geqslant 0$, we deote $z_{i+k}=z_j$.) We then call $N_{i,k}$ the complex B-spline of order k.

Evidently, $N_{i,n+1}$ is a polynomial complex spline of degree n, i.e., $N_{i,n+1} \in \mathscr{S}_n(\Delta)$.

2. Some Properties of Complex B-Splines

There are some important properties of comples B-splines; we list them below.

[P1]
$$\begin{aligned} N_{i,1}(z) &= 1, \qquad z \in I_{i,i+1}, \\ &= 0, \qquad z \in \Gamma \setminus I_{i,i+1}. \end{aligned} \tag{1}$$

$$\begin{aligned} N_{i,n+1}(z) &= \sum_{j=i}^{i+n+j} C_{ij}(z-z_j)_+^n, \qquad C_{ii} \neq 0, \quad z \in I_{i,i+n+1}, \\ &= 0, \qquad\qquad\qquad\qquad z \in \Gamma \setminus I_{i,i+n+1}, \end{aligned} \tag{2}$$

$$N_{i,n+1}(z) = \frac{z - z_i}{z_{i+n} - z_i} N_{i,n}(z) + \frac{z_{i+n+1} - z}{z_{i+n+1} - z_{i+1}} N_{i+1,n}(z). \tag{3}$$

[P2] $\{N_{i,n+1}\}_{i=j}^{j+r}$ are linearly independent on $I_{j+n,j+r+1}$ for $r \geqslant n$.

[P3] $\{N_{i,n+1}\}_{i=1}^{M}$ forms a basis of $\mathscr{S}_n(\Delta)$.

[P4] Given $A_1, \ldots, A_M$ $(=A)$, define

$$\begin{aligned} A_i^{[j]}(z) &= A_i, \qquad\qquad j=0, \\ &= \frac{z - z_i}{z_{i+n+1-j} - z_i} A_i^{[j-1]}(z) \\ &\quad + \frac{z_{i+n+1-j} - z}{z_{i+n+1-j} - z_i} A_{i-1}^{[j-1]}(z), \qquad j>0, \quad j=0,\ldots,n. \end{aligned} \tag{4}$$

If $S(z) = \sum_{i=1}^{M} A_i N_{i,n+1}(z)$, then $S(z) = \sum_{i=1}^{M} A_i^{[j]}(z) N_{i,n+1-j}(z)$. If $j=n$, then $S(z) = A_i^{[n]}(z)$, $z \in I_{i,i+1}$. We stipulate $A_0^{[j]} = A_M^{[j]}$, $j=0,\ldots,n$.

[P5]
$$\sum_{i=1}^{M} \xi_i^{(\nu-1)} N_{i,n+1}(z) = z^{\nu-1}, \qquad \nu = 1,\ldots,n+1, \tag{5}$$

where $\xi_i^{(0)} \equiv 1$, $i=\overline{1,M}$, $\xi_i^{(\nu-1)} = (-1)^{\nu-1}((\nu-1)!/n!)\, \Psi_i^{(n+1-\nu)}(0)$ and $\Psi_i(z) = (z - z_{i+1}) \cdots (z - z_{i+n})$.

The proof of these properties is similar to that for the real case [5, 8].

3. Results

We assume that the length of each interval $I_{i,i+n+1}$ is less than π: $|I_{i,i+n+1}| < \pi$ $(i=\overline{1,M})$, and that $z_{i+M} = z_i$ $(i=\overline{1,M})$.

LEMMA 1. (a) $N_{s,n+1}(z) \neq 0$, $z \in I_{s,s+n+1}$.

(b) $|N_{s,n+1}(z)| \leqslant 2^n$, $z \in \Gamma$.

Proof. Let $z' = ze^{i\theta}$, $z_j' = z_j e^{i\theta}$, $j = \overline{1, M}$, where θ is a real number $0 \leqslant \theta \leqslant \pi$. If $\tilde{N}_{s,n+1}(z')$ denotes a complex B-spline of degree n with knots $z_s', ..., z_{s+n+1}'$, we have $N_{s,n+1}(z) = \tilde{N}_{s,n+1}(z')$. We thus conclude, the values of a complex spline do not change under a rotation. Therefore, we may suppose that the point $z = 1$ does not belong to the support of $N_{s,n+1}(z)$, i.e., $1 \in \Gamma \backslash I_{s,n+1+s}$.

Under the transformation $z = (x - i)/(x + i)$, we obtain

$$N_{s,n+1}(z) = B_{s,n+1}(x) \prod_{k=s+1}^{s+n} (x_k + i)/(x + i)^n, \qquad i = \exp\left(i \frac{\pi}{2}\right), \tag{6}$$

where $B_{s,n+1}(x)$ is a real B-spline of degree n with real knots $\{x_j\}_{j=s}^{s+n+1}$, $x_j = i((1 + z_j)/(1 - z_j))$. From (6) we obtain (a).

By [P1], (3) and the information in the introduction we have (b). Q.E.D.

Hereafter, $\bar{I}_{a,b}$ denotes the closure of $I_{a,b}$.

From Lemma 1(b) and the definition of the complex B-spline we have the following:

LEMMA 2. *For* $z \in I_{i,i+n+1}$, *we have*

$$|N_{i,n+1}^{(r)}(z)| \leqslant \frac{2^n n!}{(n-r)!\, \eta_{i+n-1} \eta_{i+n-2} \cdots \eta_{i+n-r}}, \qquad r = \overline{1, n},$$

where

$$\eta_{i+n-\mu} = \min_{0 \leqslant \nu \leqslant \mu} |z_{i+\nu+n+1-\mu} - z_{i+\nu}|, \qquad 1 \leqslant \mu \leqslant r, \quad r \leqslant n.$$

Now define an operator $\mathscr{L}$ as follows. Its domain is $C^{n-1}(\Gamma)$ and it satisfies the following two conditions:

$$\begin{aligned} \mathscr{L}(g) &= \sum_{j=1}^{M} (L_j g) N_{j,n+1}, \qquad \forall g \in C^{n-1}(\Gamma), \\ L_j(g) &= \sum_{r \leqslant n} T_{j,r} g^{(r)}(t_j), \qquad j = \overline{1, M}, \quad t_j \in I_{j,j+n+1}, \end{aligned} \tag{7}$$

where $T_{j,r}$ are constants, and $g^{(r)}$ is the rth derivative of g with respect to z.

Following the idea described in the real case [5, 8], we have the following important and basic theorem about the operator $\mathscr{L}$:

THEOREM 1. *Let $\mathscr{L}$ be the operator defined by* (7). *If any one of the following three propositions* (A), (B), (C) *is valid, then the other two are true.*

(A) $\mathscr{L}(S)=S$, *for any* $S\in\mathscr{S}_n(\Delta)$.

(B) $L_j(N_{i,n+1})=\delta_{ij}$, $i,j=\overline{1,M}$. (δ_{ij} *is Kronecker delta.*)

(C) $T_{j,r}=((-1)^{n-r}/n!)\,\lambda_j^{(n-r)}(t_j)$, $r=\overline{0,n}$, $j=\overline{1,M}$, *with* $\lambda_j(z)=\prod_{k=j+1}^{j+n}(z_k-z)$, $t_j\in I_{j,j+n+1}$.

The proof of this theorem is similar to that in [5, 8] (see [3]).

Hereafter, we stipulate that the operator $\mathscr{L}$ of the form (7) satisfies one of the three conditions (A), (B) and (C), hence all.

COROLLARY. $\mathscr{L}(P)=P$, *for all* $P\in\pi_n$:= *the family of polynomials of degree n.*

We now study the error of the quasiinterpolation procedure.

Let $E=\mathscr{L}(f)-f$. $Y_zf=\sum_{r=0}^{n}f^{(r)}(z)(\cdot-z)^r/r!\in\pi_n$, $f=Y_zf+R_z$. Evidently, we have

$$E^{(s)}(z)=\frac{d^s(\mathscr{L}(R_z))}{dz^s},\qquad 0\leqslant s\leqslant n. \tag{8}$$

In the complex case, we should use the integral representation for the remainder $R_z(t)$ instead of the Lagrange formula. We then have

$$R_z^{(r)}(t)=\frac{1}{(n-1-r)!}\int_z^t R_z^{(n)}(\eta)(t-\eta)^{n-1-r}\,d\eta\qquad(0\leqslant r\leqslant n), \tag{9}$$

$f^{(n)}$—absolutely continuous on Γ

or

$$R_z^{(r)}(t)=\frac{1}{(n-r)!}\int_z^t R_z^{(n+1)}(\eta)(t-\eta)^{n-r}\,d\eta\qquad(0\leqslant r\leqslant n), \tag{10}$$

$f^{(n+1)}$—continuous on Γ.

If the arc length $|\widehat{zt}|$ is less than π, by geometry we have

$$|\widehat{zt}|<\frac{\pi}{2}|z-t|. \tag{11}$$

We suppose all the arc lengths $|\bar{I}_{i,i+n+1}|$ are less than π, $i=\overline{1,M}$. From Lemmas 1 and 2, (8), (9), (10), (11) we have:

THEOREM 2. *Let $f^{(n)}$ be absolutely continuous on Γ. Let $\mathscr{L}$ be the operator defined by* (7) *satisfying one of the three conditions* (A), (B) and (C). *Let $t_j \in \bar{I}_{j+\lambda, j+n+1-\lambda}$, for $\lambda = \lfloor (n+1)/2 \rfloor$. If $E := \mathscr{L}(f) - f$, then*

$$\begin{gathered} |E^{(s)}(z)| \leqslant K_s \omega(f^{(n)}; |\Delta|) |\Delta|^{n-s}, \qquad 0 \leqslant s \leqslant n, \\ |\Delta| = \max_{1 \leqslant j \leqslant M} |z_{j+1} - z_j|, \qquad \omega(g, h) = \sup_{\substack{|t_1 - t_2| \leqslant h \\ t_1, t_2 \in \Gamma}} |g(t_1) - g(t_2)|. \end{gathered} \tag{12}$$

If $s \leqslant \lfloor (n+1)/2 \rfloor$, then K_s is a constant independent of the mesh ratio β (see (14′)).

THEOREM 3. *If $f^{(n)}$ satisfies a Lipschitz condition of order α* $(0 < \alpha \leqslant 1)$, $|f^{(n)}(z_1) - f^{(n)}(z_2)| < D |z_1 - z_2|^\alpha$, *then*

$$|E^{(s)}(z)| < J_s |\Delta|^{n+\alpha-s}, \qquad 0 \leqslant s \leqslant n. \tag{13}$$

If $s \leqslant \lfloor (n+1)/2 \rfloor$, then J_s is a constant independent of mesh ratio.

THEOREM 4. *If $f^{(n+1)}$ is continuous on Γ, then*

$$|E^{(s)}| \leqslant P_s \|f^{(n+1)}\|_\infty |\Delta|^{n+1-s}, \qquad 0 \leqslant s \leqslant n. \tag{14}$$

If $s \leqslant \lfloor (n+1)/2 \rfloor$, P_s is a constant independent of mesh ratio.

We now estimate the constants K_s, J_s and P_s in (12), (13) and (14), respectively.

Let

$$\beta := \max_{1 \leqslant j \leqslant M} |z_{j+1} - z_j| / \min_{1 \leqslant j \leqslant M} |z_{j+1} - z_j|. \tag{14′}$$

From (8), (9), (10), (11) and Lemma 2, through elaborate calculations we have:

THEOREM 5. *The numbers K_s, J_s, P_s in Theorems* 2, 3, 4 *can be estimated as follows. For $s \leqslant \lambda$,*

$$K_s < \frac{\pi}{2} \left[\frac{\pi}{2} \left(\frac{n+2}{2} \right) + 1 \right] C_{s,1}, \tag{15}$$

$$J_s < \frac{\pi}{2} \left(\frac{n+2}{2} \right)^\alpha C_{s,1}, \tag{16}$$

$$P_s < \frac{\pi}{4n} (n+2) C_{s,1}, \tag{17}$$

while for $s > \lambda$, we have

$$K_s < \frac{\pi}{2}\left[\frac{\pi}{2}\left(\frac{n+2}{2}\right)+1\right]C_{s,2}, \tag{18}$$

$$J_s < \frac{\pi}{2}\left(\frac{n+2}{2}\right)^{\alpha} C_{s,2}, \tag{19}$$

$$P_s < \frac{\pi}{4(n+1)} C_{s,2}, \tag{20}$$

where

$$C_{s,1} = (n+1)\,n[n(n+1-s)]^{n-s}\,2^{n+s}/(n-s)!, \tag{21}$$

$$C_{s,2} = (n+1)^2(\tfrac{3}{2}n+1-s)^n\,2^n\beta^s/[(n-s)!\,(n+1-s)^s], \tag{22}$$

where β is the mesh ratio (see (14′)), α is the exponent in the Lipschitz condition (Theorem 3).

COROLLARY. *If n is fixed, let $|\Delta| \to 0$; then $\mathscr{L}(f)$ converges uniformly to f on Γ.*

REFERENCES

1. J. H. AHLBERG, E. N. NILSON, AND J. L. WALSH, Complex polynomial splines on the unit circle, *J. Math. Anal. Appl.* **33** (1971), 234–257.
2. H.-L. CHEN, Complex spline functions, *Sci. Sinica* **24**, No. 2 (1981).
3. H.-L. CHEN, "Interpolation and Approximation on the Unit Circle Part 1," Tech. Report Math. and Comp. No. 5/80. ISBN 82-7151-035-5.
4. H.-L. CHEN, "Complex Harmonic Splines. Interpolation and Approximation on the Unit Circle Part 2," Tech. Report Math. and Comp. No. 3/81, ISBN 82-7151-039-8.
5. C. DE BOOR AND G. J. FIX, Spline approximation by quasiinterpolants, *J. Approx. Theory* **8** (1973), 19–45.
6. L. FEJÉR, "Über Interpolation," pp. 66–91, Göttingen Nachr., Göttingen, 1916.
7. D. GAIER, Integralgleichungen erster Art und konforme Abbildung, *Math. Z.* **47** (1976), 113–129.
8. T. LYCHE AND L. L. SCHUMAKER, Local spline approximation methods, *J. Approx. Theory* **15** (1975), 294–325.
9. C. A. MICCHELLI AND A. SHARMA, "Spline Functions on the Circle: Cardinal *L*-Splines Revisited," MRC Technical Summary Report No. 1918 (1979).
10. L. REICHEL, "On Polynomial Approximation in the Complex Plane with Application to Conformal Mapping," TRITA-NA-8102, Dept. of Computer Science, Royal Institute of Technology, Stockholm.
11. I. J. SCHOENBERG, On polynomial spline functions on the circle, 1 and 2, *in* "Proceedings of the Conference on Constructive Theory of Functions, Budapest" (G. Alexits and S. B. Steckin, Eds.), pp. 403–433 (1972).